On the Dynamics of Cascading Failures in Interdependent Networks

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Abstract

Cascading failures in interdependent networks have been investigated using percolation theory in recent years. Here, we study the dynamics of the cascading failures, the average and fluctuations of the number of cascading as a function of system size N near criticality. The system we analyzed is a pair of fully interdependent Erdös-Rényi (ER) networks. We show that when p is close to p_c, the whole dynamical process of cascading failures can be divided into three time stages. The giant component sizes in the second time stage, presented by a plateau in the size of giant component, have large standard deviations, which cannot be well predicted by the mean-field theory. We also investigate the standard deviation of the total time $\mathbf{std}(\tau)$ using simulations. When $p = p_c$, our numerical simulations indicate that $\mathbf{std}(\tau) \sim N^{1/3}$, which increases faster than the mean, $< \tau > \sim N^{1/4}$, predicted by the mean-field theory. We also find the scaling behavior as a function of N and p of $< \tau >$ and $\mathbf{std}(\tau)$ for $p < p_c$.

1 Introduction

It has been shown that many complex systems in the real world are interdependent and interact with each other [1, 2, 3, 4]. The cascading process of failures in two fully interdependent networks has been investigated by Buldyrev *et al*. using both theory and simulations [5]. It has been shown that when a fraction $1 - p$ of nodes are randomly removed in one network, assuming that only the giant component of each network still functions, it yields cascading failures due to interdependency and percolation processes. The size of the final mutual giant component collapses abruptly as in a first order phase transition when reducing p [5, 6]. Leicht and D'Souza also provided a theoretical framework to study the percolation in a system of interacting networks [7].

Several further studies on percolation of interdependent networks have been performed. For example, Parshani *et al*. have investigated the effects of the coupling strength q between the networks (the fraction of interdependent nodes in both networks) on the phase transition properties [8]. It is found that there is a critical dependency q_c above which the transition is first order and below it is continuous as in second order phase transitions [8]. Shao *et al*. have extended the theoretical analysis to multiple support-dependent links [9]. Hu *et al*. have studied the case where both interconnectivity links and interdependency links exist [10]. Gao *et al*. generalized the study of a pair of interdependent networks to a network of coupled networks (NON) [6, 11]. Huang *et al*. and Gao *et al*. obtained analytical results for fully and partially interdependent networks under target attacks, respectively [12, 13]. Very recently, percolation on interdependent lattices has been also investigated with the finding that the vulnerability of spatially embedded interdependent networks is significantly higher compared to random networks [14, 15].

In these studies, researchers mainly focus on theoretical results based on the mean-field (MF) approach, which assumes that the number of nodes $N \to \infty$, and compare them with simulations on systems of size N. Buldyrev *et al*. analysed the effects of N on the average total time (the total number of cascades) [5]. In this paper, we mainly discuss the stages in the cascading process, and study the variations of the total time to collapse for networks of size N for fully interdependent Erdös-Rényi (ER) networks ($q = 1$).

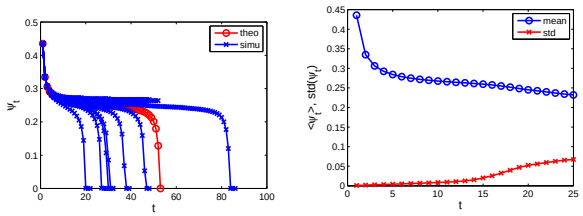
2 Three Stages in the Dynamical Process

The model of fully interdependent ER networks is defined as follow. Assume that A and B are two ER networks of the same size N. The average degrees of A and B are k_A and k_B. Each A-node a_i depends on exactly one randomly-chosen B-node b_j, and b_j also only depends on a_i. The initial attack is randomly removing a fraction $1 - p$ of A-nodes. As nodes and edges are removed, each network breaks up into connected components (clusters). Assume that when the network is fragmented, the nodes belonging to the largest component (giant component) connecting a finite fraction of the network are still functional. Since each network is connected differently, the nodes that become nonfunctional on each step are different for both networks. This will cause a cascading process of failures in the system.

First, we investigate the variations of the giant component size ψ_t of network A at time step t during the dynamical cascading failure process. Fig. 1(a) shows the giant component of network A in both theory and simulation when p is close to but below p_c.

When considering fifteen realizations in simulations in Fig. 1(a), we can see that they fall into two types according to whether the system totally collapses ultimately. Only in six realizations, the system fully collapses, and the total time τ for each of them to collapse is not the same. The dynamical process for these 6 realizations can be divided into three stages as follow: in Stage 1, the giant component decreases relatively fast after the initial attack; in Stage 2 (a plateau), the giant component size decreases very slowly; in Stage 3, the system goes toward a total collapse relatively fast. For the other nine realizations where the system does not totally collapse, we can also observe a similar Stage 1, and the dynamical process ends in Stage 2 with a non-zero mutual giant component.

Fig. 1(b) illustrates how the standard deviation of the giant component size ψ_t



(a) (b)

Figure 1: **a**. Dynamical process of the giant component size ψ_t for fully coupled ER networks in both theory and simulation (15 realizations). $k = 5$, $p = 0.4908$ for theory, $p = 0.49108$ for simulation, $N = 250,000$. **b**. Mean and standard deviation of the giant component size in the first several time steps for coupled ER networks. $k = 5$, $p = 0.49108$, $N = 500,000$, $M = 10,000$.

varies with t in the first time steps for $M = 10,000$ realizations.

While in Stage 1 we find very small variations, in Stage 2, we find large variations of the giant component sizes between different realizations. This means that the MF theory cannot well predict how long in time (how many iterations of the cascading failures) appear in Stage 2.

3 Variations of τ

As discussed in the previous section, the fluctuations of Stage 2 cannot be predicted by the MF theory. This is the feature of the fluctuations of the total time τ, at which different realizations of the system completely collapse. Here, we mainly investigate the role of p and N on the standard deviation of τ. Buldyrev *et al*. analyzed the scaling behavior of the average total time $< \tau >$ [5]. For systems of size N with $p < p_c$, it was shown that the mean of τ diverges at p_c as $< \tau > \sim 1/\sqrt{p_c - p}$; for $p = p_c$, $< \tau > \sim N^{1/4}$; for $p > p_c$, $< \tau > \sim \ln N/\sqrt{p - p_c}$.

We wish to consider here the fluctuations in τ. In our simulation, we consider the case where $p \leq p_c$, and only those realizations which totally collapses. We wish to better understand how N and p influence the mean and the standard deviation of the total time τ of the totally collapsed systems.

Fig. 2(a) and 2(b) show the effects of N and p (near p_c) on the mean and the standard deviation of τ in simulation respectively. It can be seen from Fig. 2(a) that $< \tau >$ increases with N as $N \sim N^{1/4}$ when $p = p_c \approx 0.491082$ [5]. However, when $p < p_c$, $< \tau >$ becomes a constant for larger values of N. According to these behaviors, we assume the following scaling function,

$$< \tau > \sim N^{1/4} \cdot f(u). \tag{1}$$

Here, $u = (p_c - p) \cdot N^{1/\alpha}$, and $f(u)$ is a function of u that satisfies: $f(u)$ is a constant for $u << 1$, and $f(u) \sim u^{-\alpha/4}$ for $u >> 1$.

Fig. 2(c) is a scaled version of Fig. 2(a) by plotting $< \tau > /N^{1/4}$ versus $u = (p_c - p) \cdot N^{1/\alpha}$, which aims to test the assumption of the scaling form of Eq. 1. Here, two more values of p close to p_c are included: $p = 0.4908$ and $p = 0.491$.

We find that the best choice of α for obtaining a good scaling is $\alpha = 2$. Thus, Fig. 2(c) supports the scaling form of Eq. (1). In this way, we can see that the slope of each curve changes from 0 to about $-1/2$ at $u = (p_c - p) \cdot N^{1/2} \sim 1$. Therefore, the scaling behavior of $< \tau >$ is for $N > N^* \sim (p_c - p)^{-2}$,

$$< \tau > \sim N^{1/4} \cdot u^{-1/2} = (p_c - p)^{-1/2}, \tag{2}$$

independent of N (Fig. 2(a)), and for $N \leq N^*$,

$$< \tau > \sim N^{1/4}, \tag{3}$$

independent of p (Fig. 2(a)). Note that Eqs. 2 and 3 are consistent with those of Buldyrev *et al*. [5]. Here, we suggest the full scaling behavior, Eq. 1 which yields the crossover N^*,

$$N^* \sim (p_c - p)^{-\alpha} = (p_c - p)^{-2}. \tag{4}$$

between the critical behavior for $N < N^*$ and non-critical for $N > N^*$. As seen for $p \to p_c$, $N^* \to \infty$ and for all N one observe a critical behavior.

Fig. 2(b) illustrates the effects of N and p on the standard deviation $\mathbf{std}(\tau)$. First, our numerical simulations suggest that for $p = p_c$, the slope of $\mathbf{std}(\tau)$ with N is about $1/3$, which means $\mathbf{std}(\tau) \sim N^{1/3}$, i.e., it increases faster than the mean, at least in this range of N. Furthermore, when $p < p_c$, $\mathbf{std}(\tau)$ converges for larger N, and the slope is about $-2/3$. Similarly to $< \tau >$, we assume the scaling form

$$\mathbf{std}(\tau) \sim N^{1/3} \cdot g(u), \tag{5}$$

where $u = (p_c - p) \cdot N^{1/\alpha}$, and $g(u)$ satisfies: $g(u)$ is a constant for $u << 1$, and $g(u) \sim u^{-\alpha}$ for $u >> 1$.

Fig. 2(d) shows the scaling behavior of $\mathbf{std}(\tau)$ assumed in Eq. 5 is supported by simulations with the best choice of α again about 2. The slope of the right tail in Fig. 2(d) is about -2. Therefore, for $N > N^* \sim (p - p_c)^{-2}$, we have

$$\mathbf{std}(\tau) \sim N^{1/3} \cdot u^{-2} = (p_c - p)^{-2} \cdot N^{-2/3}, \tag{6}$$

and for $N \leq N^*$,

$$\mathbf{std}(\tau) \sim N^{1/3}, \tag{7}$$

both consistent with Fig. 2(b).

Our results indicate that although the value of p_c in simulation converges to the theoretical value, the total time τ caused by complete collapses at $p = p_c$ does not converge, and varies from one realization to another. In order to better understand this behavior, we performed the following test on the total time τ in theory. First, we obtain the distribution of p_c (denoted by p_c^S to avoid confusion with p_c in theory) in simulations for each value of N. Fig. 3 shows the mean and the standard deviation of p_c^S in $3,000$ realizations for different values of N. Notice that here, the number of initially randomly removed nodes is fixed at $p = p_c$. We find that $< p_c^S >$ is almost a constant, and $\mathbf{std}(p_c^S) \sim N^{-0.49}$. This result reflects the randomness that results from the random initial attack as well as the random network structure. By linear fitting on both $< p_c^S >$ and $\mathbf{std}(p_c^S)$ respectively, we can predict the distribution of p_c^S assuming a normal distribution for any system size N. Next, we artificially use this distribution of p_c^S as the distribution of p in the initial attack for the theoretical dynamical process (Eq. 7 in SI of [5]). In this

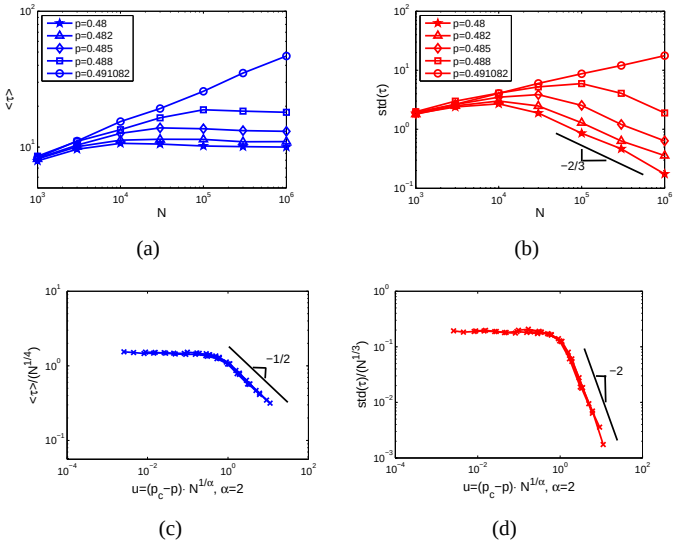


Figure 2: **a**. Effects of the system size N on the average total time τ for different values of p. $k = 5$, $M = 3,000$. **b**. Effects of the system size N on the standard deviation of the total time τ for different values of p. $k = 5$, $M = 3,000$. **c**. Scaling behavior of the average τ. Two more values of p are included: $p = 0.4908$ and $p = 0.491$. **d**. Scaling behavior of the standard deviation of τ. Two more values of p are included: $p = 0.4908$ and $p = 0.491$.

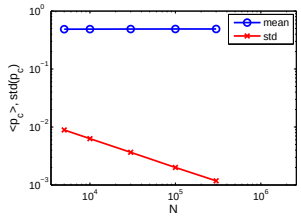


Figure 3: (Color online) Mean and standard deviation of p_c^S in simulation. $k = 5$, $M = 3,000$. The number of initially attacked nodes is fixed.

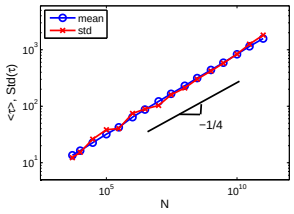


Figure 4: (Color online) Mean and standard deviation of τ in theory. $k = 5$, $M = 10,000$. For all values of N, use linear fitting on the mean and the standard deviation of p_c^S obtained in simulation, respectively, and p is chosen according to a normal distribution using these extended means and standard deviations.

way, we introduce randomness into the theory. Fig. 4 shows that in this modified theoretical process, both $< \tau >$ and $\mathbf{std}(\tau)$ increase as close to $N^{1/4}$. Compared with our simulation results (Fig. 2(b)), this behavior indicates that apart from the random initial attack, other sources that cause more randomness on the total time should exist, like the network topologies and the critical behavior in Stage 2.

Acknowledgments

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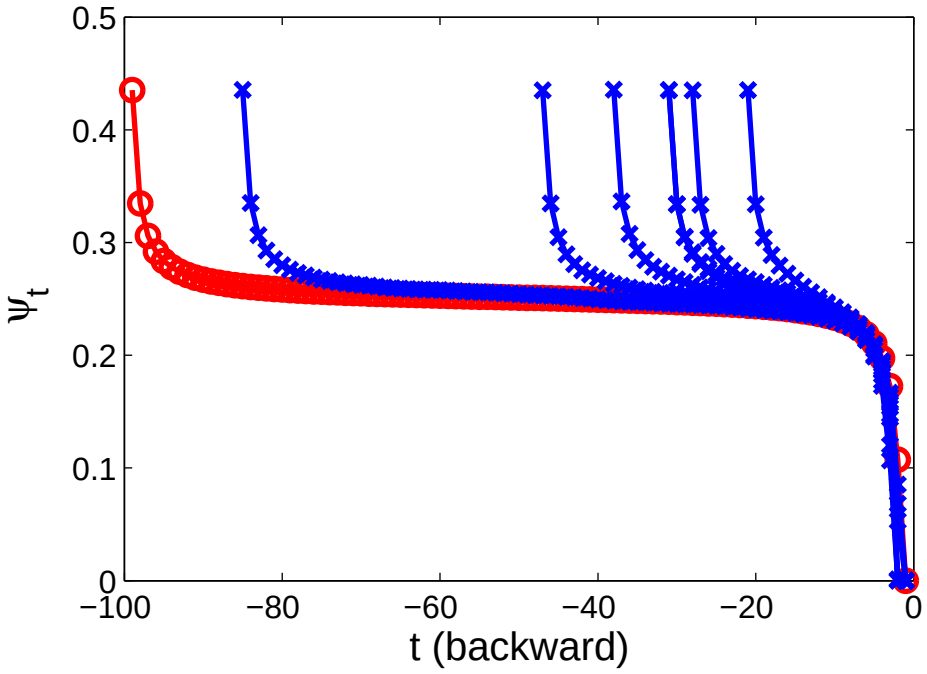
0.5
0.4
0.3
0.2
0.1
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ψ_t
−100
−80
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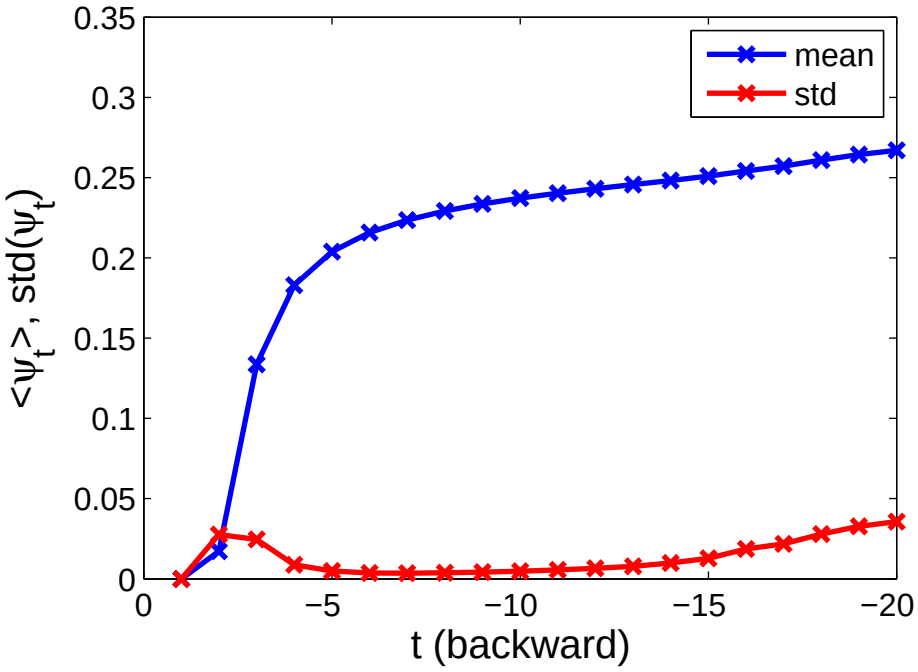
t (backward)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